

Bordism Field Theories I

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November 13, 2024

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May 2024

Abstract

This is the first in a series of articles explaining the relationship between (co)bordisms and (topological quantum) field theories.

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1 Introduction

In recent times, it has become fashionable to think of a *field theory* as a (symmetric monoidal) functor $\mathcal{Z} : \mathbf{B} \rightarrow \mathbf{T}$ where $\mathbf{B}$ is a suitable bordism category and $\mathbf{T}$ is the target category; namely, $\mathcal{T}$ is either defined to be the category $\mathbf{Vect}_k^{\otimes}$ of tensorial vector spaces with ground field k , as in [1] and [12], or the category $\mathbf{TV}$ of *topological* vector spaces, as was pursued by Stolz and Teichner in [2]. The first has the advantage of describing a common principle in QM: that the state space of two independent systems is the tensor product of those systems; meanwhile, the second is more correct for certain subtle mathematical reasons: namely, in order for our theory to be a truly *topological* field theory, it must take place in this category.

Inklings of this paradigm can be found in the early works of Witten, Atiyah, Kontsevich, but most importantly Segal, who, in 1988 established that if $\Phi(\mathcal{M})$ is the genus of a manifold $\mathcal{M}$, then two mutually cobordant manifolds $\mathcal{M}_0$ and $\mathcal{M}_1$ are such that $\Phi(\mathcal{M}_i)$ is a constant for $i \in \langle 1 \rangle$, where $\langle n \rangle$ is the set $\{0, \dots, n\}$. We will say, following [15], that two cobordisms are equivalent if

they have the same genus, and the same number of input-output boundaries. Please refer to the pictures in that reference, as this concept is perhaps best understood pictorially, as opposed to e.g. algebraically, although the author of this article is not equipped to provide such illustrations.

The basic machinery is this: you have a d -dimensional manifold $\mathcal{B}$ consisting of disjoint $d - 1$ -dimensional submanifolds $\mathcal{Y}, \mathcal{Y}'$, etc. (usually just the two), and define a category $\mathbf{d}\text{-Bord}$, which reduces to a category $\mathbf{E}\text{-Bord}$ in the Euclidean case. Then, given n submanifolds, we can identify the disjoint union of all of them with an n -ary product operation. Observe:

$$\prod_{i=0}^n \mathcal{Y}_i = \mathcal{Y}_0 \sqcup \dots \sqcup \mathcal{Y}_n$$

$\mathbf{d}\text{-Bord}$ is a “category” in the sense that, for a given index set $\mathcal{I}$, a $\min(i \in \mathcal{I})$ -dimensional submanifold becomes an object, and a $\min(i) + 1$ -dimensional submanifold becomes an object in the path category over $\mathcal{B}$, which for our purposes means that it is a morphism between bordant submanifolds. We are allowed, following Stolz and Teichner [2] to substitute d -dimensional manifolds for $d|b$ -dimensional ones; in other words, we are free to superize.

To a co-dimension one slice of $\mathcal{B}$ (which may be chosen to represent the whole of spacetime, as in [15]), the field theory functor¹ $\mathcal{Z}$ assigns the Hilbert space of quantum states to every manifold in the equivalence class $[\mathcal{Y}]$, and to $\mathcal{B}$ it assigns a scattering amplitude²: $\mathcal{Z}(\partial_{in}\mathcal{B}) \rightarrow \mathcal{Z}(\partial_{out}\mathcal{B})$. When we apply the functor to the empty set, we recover our ground field k , along with a trivial scattering amplitude, which manifests as a sampling of the wavefunction of a particle, where by “wave function” $\Psi(p)$, we mean a collection of oscillatory modes for the worldline of p , which are constrained by one’s choice of gauge.

The time evolution operator $\mathcal{T} : |\psi(t)\rangle \text{ to } |\psi(t + \varepsilon)\rangle$ is given by decomposing the state space assigned to each $(d-1)$ -dimensional of $\mathcal{B}$ into a topos/lattice; i.e., by ordering the Hilbert space in such a way that there is a bottom element $(0_{\mathcal{H}})$ and a top element, $(1_{\mathcal{H}})$, both of which are (topological) vector spaces. The elements are then paired together so that the following diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{Ric_{g_{i+1}}} & \mathcal{Y}' \\ \downarrow \mathcal{Z} & & \downarrow \mathcal{Z} \\ 0_{\mathcal{H}} & \xrightarrow{\mathcal{T}} & 1_{\mathcal{H}} \end{array}$$

commutes, where $Ric_{g_{i+1}}$ is our Ricci iteration, and $\mathcal{Z}$ is then our field theory. In other words, our bordism category is sent to separable witnesses of the category of finite co-dimension 1 Hilbert spaces, $FD\mathbf{Hilb}_{d-1}^1$, and the identity type of a causal moment is given by forcing these witnesses to be equivalent.

Notation 1.1. *We may use the terms “cobordism” and “cobordism class” interchangeably, as maps in the category $n\mathbf{Cob}$ are technically cobordisms themselves.*

¹Also called the “quantization functor” in [12]

²In [15], these are (confusingly) called “transition amplitudes,” or “Feynman path integrals.” We shed some light on this later; for now though, each morphism in our bordism category should be thought of as a 1-form; namely, the vectors that it eats are gauge potentials, and the scalar that it spits out is the field strength; thus, for a path $\gamma \in n\mathbf{Bord}$, we have that $\gamma = \int_{\gamma} \frac{\partial x_i}{\partial t}$, where x_i is the momentum of a particle x ; i.e., a scalar assigned piecewise linearly to each spacetime/worldsheet vector over x

The most general of the bordism categories is the category of $\mathcal{G}$ -equivariant bordisms, $\mathcal{G} - \mathbf{Bord}$ as defined by Ludewig-Stoffel in [3]. This is a very nice category, but when generalizing to such an extent, there are several nuances one must take into consideration. For instance, we usually work with *isomorphism classes* of submanifolds of $\mathcal{B}$ rather than the submanifolds themselves. Each of these isomorphism classes are associated with an isomorphism class of *metrics* over the class $[\mathcal{Y}]$ of submanifolds, and as is noted in [1], these may not be glued. Even more technically, in terms of the underlying model structure of $\mathcal{G} - \mathbf{Bord}$, there may exist problematic maps $o \rightarrow o'$, with o a fibrant object and o' non-fibrant.

1.1 1-Bord

The simplest possible category to work with is the category 1-Bord , which takes non-trivial 1-dimensional curves to be our basic objects. Write γ for such a curve. Then, we take the map $\partial\gamma \rightarrow \mathcal{B}$, where $\partial\gamma$ is an irreducible representation of the boundary component of the curve. This is actually a rather curious object because, even though it is by definition zero-dimensional, it is spatially extended. In order to account for this, we need the notion of an isotropy group of a *formal* point. Many facts are known about these groups when they act on foliations; see for instance [6]. To summarize, $Iso(\partial\gamma)$ will give rise to a normal subgroup $Fix(\partial\gamma)$ of biholomorphisms sending the boundary component to itself. Now, $Fix(\partial\gamma)$ actually defines the germ of a collar of a path; as per [1], these, *together* with the points of $\mathcal{B}$ form the data of an isomorphism class for a cobordism. So, the relationship $\mathcal{Y} \sim \mathcal{Y}'$ holds, for disjoint $\mathcal{Y}, \mathcal{Y}'$, whence the map $F_{\mathcal{Y}} \rightarrow F_{\mathcal{Y}'}$ is an equivalence, where $F_{\mathcal{Y}}$ is defined to be $Fix(\partial\gamma) \in \mathcal{Y}$, $F_{\mathcal{Y}'}$ is defined to be the corresponding object in $\mathcal{Y}'$, and the orbits of any neighborhood $\mathcal{U}(\gamma_i)$ with $i \in \gamma$ are equivalent. This is actually a rather strong notion of equivalence, but it is still relatively weak enough that we can construct the family $Fam(\mathcal{B}, \sim)$ of cobordisms modulo this relationship, without sacrificing generality.

The category 1-Bord consists of, to summarize: a collection of curves $x_i \rightarrow x_j$ for every $j > i$, and maps $x_j^b : x_{\langle j \rangle} \rightarrow X$ for some compact manifold $X \in \mathcal{B}$ as morphisms. We then identify X with a fundamental ∞ -groupoid, and in the words of Lurie [7], X is the classifying space for 1-Bord ; that is to say, the path components of X are in bijection with the objects of 1-Bord .

Proposition 1.1. $\partial\gamma = Fix(\gamma)$

Proposition 1.2. $Map(\partial\gamma, X) \cong B1\text{-Bord}$

Proof. Follows from Lurie's definition of a classifying space, and our definition of the category 1-Bord □

Proposition 1.3. $\partial\gamma \subset \bar{\gamma}$

Proof. Since $\bar{\gamma} \supset \gamma$ and $\partial\gamma \in \gamma$ hold, then by transitivity, this is true. □

Lemma 1.1. $\partial\gamma = spec(k) \ni m\gamma + n$

Proof. Let $\gamma = [0, 1]$. Then, γ contains no prime ideals which are integers.

Let $m = 3$ and $n = 2$. Then, we gain the interval $[2, 5]$ which has as ordinary primes the set $\{2, 3, 5\}$. Construct the set $\mathcal{V}$ of Von-Neumann universes $V_2, V_3, V_5 \simeq \langle 2 \rangle, \langle 3 \rangle, \langle 5 \rangle$. Let n be the number of points in $\langle n \rangle$. Then, there is a bijection between $\mathcal{V}$ and a distinguished collection of simplicial complexes $\Delta_{[2, 3, 5]}$.

This shows that the category $\Delta(\mathcal{V})$ is triangulated. We then construct a foliation of $\mathcal{B}$ which contains every piecewise disjoint component of γ , which becomes a unit in $k^\times$; then, X classifies the isotropy group $Iso(x)$, and by a previous proposition, $\text{spec}(k) \subseteq \gamma$. Since k is defined to be the smallest extension of γ containing the closure of $m\gamma + n$, the lemma is shown to be true. $\square$

Warning 1.1. *Since one of our lemmas may fail to hold in the non-holomorphic case, our lemma does not lift to purely Euclidean field theories, and may fail to lift to more general categories.*

1.2 Morse functions

Cobordisms are smooth, and this smoothness is given from their cylindrical structure, which was demonstrated by Kock to be an extension of Hirsch's regular interval theorem [17].

Theorem 1.1 (Hirsch). *Let $M : \Sigma_0 \rightarrow \Sigma_1$ be a cobordism and let $f : M \rightarrow [0, 1]$ be a smooth map without any critical points, such that $\Sigma_0 = f^{-1}(0)$ and $\Sigma_1 = f^{-1}(1)$. Then there is a diffeomorphism from the cylinder $\Sigma \rightarrow [0, 1]$ to M compatible with the projection to $[0, 1]$ such that the following diagram:*

$$\begin{array}{ccc} \Sigma_0 \times [0, 1] & \xrightarrow{\sim} & M \\ & \searrow & \downarrow f \\ & & [0, 1] \end{array}$$

commutes.

The corollary of which is:

Theorem 1.2 (Kock). *Let $M : \Sigma_0 \Rightarrow \Sigma_1$ be a cobordism. Then, there is a decomposition $M = M_{[0, \varepsilon]} M_{[\varepsilon, 1]}$ such that $M_{[0, \varepsilon]}$ is diffeomorphic to a cylinder over Σ_0 . The same holds for the part near Σ_1 .*

The decomposition is given by taking a Morse function $f : M \rightarrow [0, 1]$ and letting t be the first critical value. Then for $\varepsilon < t$, the interval $[0, \varepsilon]$ is regular, so cutting M along the inverse image $f^{-1}(\varepsilon)$ yields the desired result by the regular interval theorem. Furthermore, we can glue cobordism classes and their Morse functions together. Given two cobordisms $M_0 : \Sigma_0 \rightarrow \Sigma_1$ and $M_1 : \Sigma_1 \rightarrow \Sigma_2$, their Morse functions $f_0 : M_0 \rightarrow [0, 1]$ and $f_1 : M_1 \rightarrow [1, 2]$ can be glued by considering $M_0 \amalg_{\Sigma_1} M_1$ as a topological space admitting a map to the interval $[0, 2]$. Then, ε is chosen to be small enough that the two intervals, $[1, 1 + \varepsilon]$ and $[1 - \varepsilon, 1]$ are regular for $f_{\langle 1 \rangle}$. Then, the inverse images of these intervals are cylinders, and thus we have a smooth structure.

2 Families on Bordisms

In order to describe a generic field theory, we actually need to replace our notion of a TQFT with that of a TQFT *family*, because for any field theory $E : \mathcal{B} \rightarrow \mathcal{X} \in \mathcal{T}$ with $\mathcal{X}$ a topological space, we really mean a *family* of field theories, $Fam_E(\mathcal{B}, \mathcal{X})$ from the chosen bordism category $\mathcal{B}$ which is parameterized by the target manifold $\mathcal{X}$. Subsequently, all of our other objects will need a homologous replacement; for instance, the notion of a vector space $\mathcal{V}$ will need to be replaced by an equivalence class $Fam(\mathcal{V}, S)$, or in otherwords, an S -family $[V/S]$ of vector spaces over S .

The same story holds for vector bundles; for a vector bundle $Bun_Y : E \rightarrow B$, we need a family $Fam(Bun_Y, S)$ such that for every fiber $f \in Bun_Y$, f is realized as the restriction of $[f/S]$ to f_θ , where θ is a Mochizuki θ -link.

The θ -link is described in [5, pg. 17], but for convenience we will summarize and say it is an anti-holomorphic reflection of a model of scheme theory. We will not need all of the technical details here; however, the reader should be aware of the following nice property of θ : that it serves two purposes. The first purpose is as follows: if we have two model categories, $\mathcal{C}_i$ and $\mathcal{C}_j$ with $j \neq i$, then $\theta\mathcal{C}_j = \mathcal{C}_i\theta = \mathcal{C}_{j \circ i}$, and analogously, by replacing j with i and vice versa, we have $\mathcal{C}_{i \circ j}$. In general, suppose we have two categories L and R ; θ is then like a $\mathbb{T}$ -torsor³ acting on L from the right, and acting on R from the left. Thus, a complete θ link is something like $L\theta R$, and we should think of this differentially; that is to say, as a map $\theta : {}_L A^{-1} \rightarrow A_R$ between charts, such that the following diagram:

$$\begin{array}{ccc} L & & R \\ \downarrow \flat & & \downarrow \flat \\ {}_L A & \xrightarrow{\theta} & A_R \\ \theta_j \uparrow & & \uparrow \theta_i \\ A_R^{-1} & & {}_L A^{-1} \end{array}$$

commutes, and we have

$$\theta \circ \theta_j = \theta \circ \theta_i = \theta$$

Let $\mathbf{Fam}$ denote the category of families of smooth manifolds. Then, as per [3], $\mathbf{Fam}$ is a stack. Furthermore, for every smooth manifold $\mathcal{P} \in \mathbf{Man}$, there is a smooth embedding $q : \mathcal{P} \hookrightarrow \mathbf{Fam}$, and for every distinct manifold $\mathcal{P}'$, there is a fiberwise open embedding $\mathcal{P}' \rightarrow \mathcal{P}$ such that $Fam_q(\mathcal{P}, S) = Fam_q(\mathcal{P}', S')$, and the following diagram

$$\begin{array}{ccc} \mathcal{P}' & \xrightarrow{F} & \mathcal{P} \\ \downarrow & & \downarrow \\ S' & \xrightarrow{f} & S \end{array}$$

commutes, and the map $\mathcal{P}' \rightarrow S' \times_S \mathcal{P}$ is an open embedding. We denote by $\mathbf{Fam}^d$ the full subcategory of families with d -dimensional fibers.

3 Extending Down

Consider a bordant d -manifold $X := \prod_{i=0}^{\infty} Y_i$ with $\dim(Y_i) = d - 1$. As per [7] and [9], we can think of X as an (∞, d) -category; i.e., the category $\mathbf{Fam}^d$, with d -dimensional fiber. Suppose this category is Vec -enriched. Then, through a series of nested inclusions, we can create a flag:

$$\mathbf{Fam}^0 \subset \dots \subset \mathbf{Fam}^{d-1} \subset \mathbf{Fam}^d$$

so that

$$\lim_{\leftarrow} Y_{\bullet} = *$$

³See [8] for more information.

Denote by Y_{ij} a d -2-dimensional submanifold of Y_i indexed by $j \in J$, with J a proper subset of I . Denote by Y_{ijk} the corresponding d -3-dimensional submanifold, and so on and so forth. For $d = \infty$, denote by $Y_\omega = Y_{ijk\dots\omega}$ a manifold of minimal dimension compatible with the family of vector bundles over X ; in other words, the smallest manifold with a fiberwise obedience to the generalized cocycle condition, and compatible with all of the restrictions on X .

Proposition 3.1. *Y_ω is simply connected.*

Proof. Since $\min(\dim(Y_\bullet)) = 0$, and since the boundary of a zero-dimensional manifold is empty, then Y_ω is given by $\coprod_{\emptyset} Y$, meaning that it is the disjoint union of copies of the empty set. This means that the category 0-Bord consists of maps $\emptyset \rightarrow \mathbf{T}$, and by convention we say that the genus of the empty set is always zero. Therefore, Y_ω is simply connected. $\square$

Remark 3.1. *Because 0-Bord is a 0-category, it consists only of objects, and not morphisms; in order for a space to have a non-zero genus, there must be an obstruction to a morphism between objects. However, no such obstructions arise when we consider only single-object categories. Therefore, this would be another way of proving the above.*

Remark 3.2. *If X admits a map $c_\infty; X \rightarrow *$; i.e., if it is contractible to a point, then it must by its nature be simply connected, as in the limit*

$$\lim_{i \rightarrow \infty} c_i(X)$$

we have $d(\alpha, \beta) \rightarrow 0$ for all distinct points α and β .

Let λx mean the eigenvalues of a matrix x . Define Λ as $\int_k \gamma(x \cdot \dots \cdot y)$, where the elided portion is a collection of objects whose Euclidean transformations are co-linear. In the ordinary quantum theory, we have that $k = \mathbb{C}$. The operator is defined pointwise on the interval $[x, y]$.

Let ∇ be a connection on a topological vector space. Suppose we have two partons, α and β on a d -dimensional orbifold $\mathcal{B}_d$. If, for some sufficiently small ε , we have $d(\alpha, \beta) \leq \varepsilon$, then we wish to observe the behavior of ∇ under the map $\xi : d \rightarrow 0$. Suppose α and β lie in the convex portion of $\mathcal{B}_d$, and further suppose they belong to an equivalence class A . Suppose further that $d(\text{sing}(\mathcal{B}_d), A) \gg \varepsilon$, where $\text{sing}(\mathcal{B}_d)$ is a vacuum-like singularity. Then, assuming $\beta = \alpha \nabla_\gamma$, the map ξ is essentially a contraction $\gamma \rightarrow \mathbb{1}$ to the one-object groupoid containing a single point at which Λ converges.

This tells us that the identity of a single quantum is not strong; rather, identity is generated by a specific choice of quantization, and when two quanta are in close enough proximity, they become indistinguishable. In other words, the resolution is too poor to make a distinction between α and β , though this problem is remedied by a change of bases $b_{\mathcal{B}_d} \rightarrow b_{\mathcal{B}_{>d}}$. This is resonant with a Kaluza-Klein style compactification $[\hat{q}] \rightarrow q$, where $[\hat{q}]$ is a class of quasi-quanta. When we “extend down,” we replace the notion of identity with a weaker equivalence, which in [9] was called the pseudo-identity; this is actually a pseudo-identity on quasi-quanta, but an identity on the single quantum generated by said quasi-quanta. We conjecture that it is impossible to derive an identity on a quasi-quantum in a 3+1-dimensional Minkowski framework, and we must extend our spacetime in order to do so.

3.1 Pseudo-identities

Write $\widetilde{Id}_X$ for a pseudo-identity on X ; we have $Id_{Y_\omega} = \widetilde{Id}_x$ where $x \subset X$ is a manifold of dimension strictly less than d . The following diagram elucidates the situation:

$$\begin{array}{ccccc}
 * & \xleftarrow{\quad} & x & \hookrightarrow & X \\
 & \searrow Id & \downarrow \widetilde{Id} & \swarrow Id & \downarrow Id \\
 & & Y_\omega & \xleftarrow{\quad} & Y_{<d}
 \end{array}$$

where $x \hookrightarrow X$ is a flat embedding, and Id_X is the trivial functor sending a boundary-less manifold to itself. Thus, $x \in \text{int}(Y_{<d})$. In the string case, we simply specify that X is spin and oriented, and we let $x = \int_X x_i$ be a string on $\mathcal{B}_d$, which can be thought of as an $NS5$ -brane. We will omit the technical details, but the reader should know that an $NS5$ -brane is a brane with boundary conditions such that we may glue an AdS_5 brane to one at the boundary. The exact gluing conditions need only be specified in the abstract for now; for future reference, they involve the construction of a quasi-coherent sheaf $QC(Y_{<d})$ of submanifolds, which, depending on our intent, may be restricted only to Lagrangian, Lorentzian, etc. submanifolds. We will define the gluing of an AdS_5 brane onto an $NS5$ -brane by the map $\mathcal{Glu}_5 : Psh(Y_{<5}) \rightarrow QC(Y_{<5})$.

We further specify that the above diagram is applicable for any map $\mathcal{Glu}_n$ for any $n \geq 1$. A further technical trick, in which we treat the desuspension of a point as a map onto the empty set, would allow us to include $\mathcal{Glu}_0$ as well. In the future, once the mathematics of negative diimensions [10] have been fully worked out, we may have a complete theory of $\mathcal{Glu}_z$ for all $z \in \mathbb{Z}$.

Remark 3.3. *A full theory of $\mathcal{Glu}_z$ would shed some light on David Bohm's "implicate and explicate order; if we let all of sheaves of negative dimensional manifolds be indexed by families of implicate operators somehow, and assign to the positive ones explicate (semi-classical) operators, we could very well develop a superalgebra involving pseudo-identities. In my mind, we let $\mathcal{Glu}_{<1>}$ be the achiral element, and we can write a torsor as a diagram where this is the principal map.*

4 Frobenius Algebras

It is well known that a 2-dimensional TQFT corresponds to a Frobenius algebra; see, e.g. [7], [14, pg. 2], and [15].

Definition 4.1 (Kock). *A Frobenius algebra is a finite-dimensional algebra equipped with a nondegenerate bilinear form compatible with the multiplication. Examples are matrix rings, group rings, the ring of characters of a representation, and Artinian Gorenstein rings (which in turn include cohomology rings, local rings of isolated hypersurface singularities, etc.)*

The above is lifted verbatim from [15]. We also reproduce the following:

Theorem 4.1. *There is an equivalence of categories*

$$2TQFT_k \simeq cFA_k$$

where the left-hand-side is a 2-dimensional topological quantum field theory, and the r.h.s. is a commutative Frobenius k -algebra.

The above equivalence is given by sending a TQFT to its value on the unique connected closed 1-manifold; i.e., the Tate circle, S_t^1 .

Remark 4.1. *We could very well state this generically, and replace the Tate circle with the classical one, but because of its prominence in motivic homotopy theory, as in [16], we use Tate's instead. This is suggestive, as it allows us to ponder the bordism category as having a pullback, $n\text{Bord}_* \cong \text{Mot}_k$, which corresponds to the category of motives over k . The situation is summarized by the following commutative diagram:*

$$\begin{array}{ccc} n\text{Bord} & \longrightarrow & \text{Vect}_k^\otimes \\ \downarrow & \nearrow & \\ \text{Mot}_k & & \end{array}$$

which we extend to

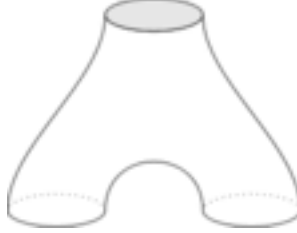
$$\begin{array}{ccccc} n\text{Bord} & \longrightarrow & \text{Vect}_k^\otimes & & \\ \downarrow & \nearrow & \downarrow t & & \\ \text{Mot}_k & \xrightarrow[t]{} & k & & \end{array}$$

by considering the trivializations of Mot_k and $\text{Vect}_k^\otimes$ given by choosing 0Bord as our base source category, which restricts us to maps out of the empty set.

Let A be a Frobenius algebra, and let

$$\Sigma_i = \begin{cases} \partial_{in}\mathcal{B} & \text{if } i = 0 \\ \partial_{out}\mathcal{B} & \text{if } i = 1 \end{cases}$$

Then, tensoring A with itself on the left-hand n -many times means we have $\text{card}(\Sigma_0) = n$, and tensoring A with itself m many times on the right-hand side means we have $\text{card}(\Sigma_1) = m$; meanwhile, tensoring with k on either side means we have a unit; i.e., a boundary sector without a collar. Thus, the following cobordism may be written $A \otimes A \rightarrow A$ whence the bottom is considered to be the input, and $A \rightarrow A \otimes A$ otherwise:



Lemma 4.1 (Monoidality). *$(n\text{Cob}, \coprod, \emptyset)$ is a monoidal category.*

This is again due to Kock, without whom the author would not even be able to write this paper.

Proof. If $\mathcal{Y}_0$ and $\mathcal{Y}_1$ are $(n-1)$ -manifolds, then the disjoint union $\mathcal{Y}_0 \coprod \mathcal{Y}_1$ is again an $(n-1)$ -manifold, and given two cobordisms $\mathcal{B} : \mathcal{Y}_0 \Rightarrow \mathcal{Y}_1$ and $\mathcal{B}' : \mathcal{Y}'_0 \Rightarrow \mathcal{Y}'_1$, their disjoint union is a natural cobordism $\mathcal{Y}_0 \coprod \mathcal{Y}'_0 \Rightarrow \mathcal{Y}_1 \coprod \mathcal{Y}'_1$. Further, the empty manifold $\emptyset_n$ is a cobordism $\emptyset_{n-1} \Rightarrow \emptyset_{n-1}$. $\square$

A Future Work

In the spirit of making things Ehresmannian⁴, we will consider cobordisms as frame bundles with principle connection given by the time step operator. Then, each submanifold of the cobordism can be treated as a frame (of reference), and the bordant componenet of the manifold will constitute a fiber space whose projection class $[\pi]$ is given by a chain of Frobenius algebras furnished with group operations.

Furthermore, there is an (over)abundance of literature on the topic at hand, but what can be surmised from it is thin in a vacuum. Therefore, as a thickener, we will mix in some other key ingredients and establish points of contact with other areas of physics; these may encompass dark energy, skyrmions, and string compactification.

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⁴See Eric Weinstein [18] for the advantages of Ehresmannian geometry